JOURNAL OF APPROXIMATION THEORY **41**, 21–28 (1984)

Quadratic and Cubic Spline Interpolation

SHENQUAN XIE*

Department of Computer Science,
State University of New York at Buffalo, Amherst, New York 14226, U.S.A.

Communicated by Oved Shisha

Received October 5, 1982; revised June 13, 1983

Some variational properties of (2, 0) and (3, 1) spline interpolations and their error estimates are considered.

1. INTRODUCTION

Let (k, m) denote the class of k degree interpolating polynomial splines whose interpolation conditions are given by mth derivative values. The variational properties of $(2k, 2m-1)$ and $(2k-1, 2m)$ have been considered by many researchers (e.g., [1, 2]), but very few have considered the variational properties of $(2k, 2m)$ and $(2k-1, 2m-1)$. In [3], Sharma and Tzimbalario considered the variational properties of some kinds of quadratic spline interpolation. Here we study the (2, 0) and (3, 1) problems. The discussion of the variational properties is an extension of [3]. The error bounds for the (2, 0) problem have been considered in [2–8]; here we shall give more precise error bounds. The discussion of (3, 1) is a direct extension of that for (2, 0).

2. INTERPOLATION PROBLEMS

For $-\infty < a < b < +\infty$ and for any positive integer $n \geqslant 2$, let

$$\Delta : a = x_0 < x_1 < \cdots < x_n = b$$

denote a partition of $[a, b]$ with knots x_i and steps $h_i = x_{i+1} - x_i$. Let $Sp(\Delta, k)$ denote the class of k degree polynomial splines on Δ.

* Present address: Xiangtan Univeristy, Xiangtan, Hunan 411175, People's Republic of China.

0021-9045/84 $3.00

(2, 0) *interpolation problem.* Find $s(x) \in Sp(\Delta, 2)$ such that

$$s(x_i) = f_i \qquad (i = 0, 1, \ldots, n),$$
$$s'(x_j) = f'_j \qquad (j = 0 \text{ or } n).$$

(3, 1) *interpolation problem.* Find $s(x) \in Sp(\Delta, 3)$ such that

$$s'(x_i) = f'_i \qquad (i = 0, 1, \ldots, n),$$
$$s(x_j) = f_j \qquad (j = 0 \text{ or } n),$$
$$s''(x_j) = f''_j \qquad (j = 0 \text{ or } n).$$

3. Functionals

Let $PC^k[a, b] = \{ g(x) \mid g(x) \in C^{k-1}[a, b],\ g^{(k)}(x)$ is piecewise continuous on $[a, b]$ and has there at most a finite number of discontinuities of the first kind$\}$,

$$PC^k_\Delta[a, b] = \{ g(x) \mid g(x) \in PC^k[a, b], g^{(k-2)}(x_i) = f_i^{(k-2)}, i = 0, 1, \ldots, n\}.$$

Consider the functionals

$$J[f^{(k-1)}] = \sum_{i=0}^{n-1} J_i[f^{(k-1)}] \qquad (k = 2, 3),$$

$$J_i[f^{(k-1)}] = \int_{x_i}^{x_{i+1}} [f^{(k-1)}(x) + f^{(k-1)}(x_i + x_{i+1} - x)]^2 \, dx,$$
$$(i = 0, 1, \ldots, n-1).$$

THEOREM 3.1. *Let $f(x) \in C^{k-1}[x_i, x_{i+1}]$. Then $f(x)$ is a solution of the functional equation $J_i[f^{(k-1)}] = 0$ if and only if*

$$f^{(k-2)}(x) = f^{(k-2)}(x_i + x_{i+1} - x), \tag{3.1}$$

i.e., on $[x_i, x_{i+1}]$ $f^{(k-2)}(x)$ is symmetric about the midpoint of $[x_i, x_{i+1}]$.

Proof. By differentiation of (3.1) we obtain

$$f^{(k-1)}(x) + f^{(k-1)}(x_i + x_{i+1} - x) = 0, \tag{3.2}$$

which implies

$$j_i[f^{(k-1)}] = 0.$$

Conversely if $J_i[f^{(k-1)}]=0$, then integrating (3.2) gives $f^{(k-2)}(x)-f^{(k-2)}(x_i+x_{i+1}-x)=c$. Setting $x=(x_i+x_{i+1})/2$, we obtain $c=0$. Q.E.D.

From Theorem 3.1 we obtain

THEOREM 3.2. *Let $f(x)\in PC^k[a,b]$. Then $f(x)$ is a solution of $J[f^{(k-1)}]=0$ if and only if* (3.2) *holds for $i=0,1,\dots,n-1$, i.e., on every $[x_i,x_{i+1}]$ $f(x)$ is symmetric about the midpoint of the interval.*

For spline functions the conditions above can be simplified.

THEOREM 3.3. *Let $s(x)\in Sp(\Delta,k)$. Then $s(x)$ is a solution of $J_i[f^{(k-1)}]=0$ if and only if*

$$s^{(k-2)}(x_i)=s^{(k-2)}(x_{i+1}).$$

Proof. The necessary condition has been given in Theorem 3.1. For the sufficient condition integrate by parts:

$$\begin{aligned}J_i[s^{(k-1)}]=&[s^{(k-2)}(x)-s^{(k-2)}(x_i+x_{i+1}-x)]\\&\times[s^{(k-1)}(x)+s^{(k-1)}(x_i+x_{i+1}-x)]\big|_{x_i}^{x_{i+1}}\\&-\int_{x_i}^{x_{i+1}}[s^{(k-2)}(x)-s^{(k-2)}(x_i+x_{i+1}-x)]\\&\times[s^{(k)}(x)-s^{(k)}(x_i+x_{i+1}-x)]\,dx=0.\end{aligned}$$

THEOREM 3.4. *Let $s(x)\in Sp(\Delta,k)$. Then $s(x)$ is a solution of $J[f^{(k-1)}]=0$ if and only if*

$$s^{(k-2)}(x_0)=s^{(k-2)}(x_1)=\cdots=s^{(k-2)}(x_n).$$

4. EXISTENCE AND UNIQUENESS

THEOREM 4.5. *The solutions of* $(2,0)$, $(3,1)$ *spline interpolation problems uniquely exist.*

Proof. It is enough to show there exists only the trivial solution for the homogeneous interpolation problem.

By Theorem 3.2, 3.4:

$$s^{(k-2)}(x_0)=s^{(k-2)}(x_1)=\cdots=s^{(k-2)}(x_n)=0,$$
$$s^{(k-1)}(x_0)=-s^{(k-1)}(x_1)=\cdots=(-1)^n\,s^{(k-1)}(x_n)=0.$$

So $s^{(k-2)}(x)\equiv 0$ $(x\in[a,b])$. If $k=3$, by $s(x_j)=0$ ($j=0$ or n), $s(x)\equiv 0$.

Q.E.D.

THEOREM 4.6. *For another* (3, 1) *problem: find* $s(x) \in Sp(\Delta, 3)$ *such that* $s'(x_i) = f'_i$ $(i = 0, 1, \ldots, n)$, $s(x_j) = f_j$ $(j = 0, n)$: *the solution uniquely exists if and only if* $h_0^2 - h_1^2 + \cdots + (-1)^{n-1} h_{n-1}^2 \neq 0$.

Proof. By Theorems 3.2, 3.4:

$$s'(x_0) = s'(x_1) = \cdots = s'(x_n) = 0,$$

$$s''(x_0) = -s''(x_1) = \cdots = (-1)^n s''(x_n).$$

Therefore,

$$s(x) = c_i - 3d_i h_i (x - x_i)^2 + 2d_i (x - x_i)^3$$

$$(x \in [x_i, x_{i+1}], i = 0, 1, \ldots, n-1),$$

$$d_0 h_0 = -d_1 h_1 = \cdots = (-1)^{n-1} d_{n-1} h_{n-1}.$$

As $s(x_0) = s(x_n) = 0$, we have

$$d_0 h_0 (h_0^2 - h_1^2 + \cdots + (-1)^{n-1} h_{n-1}^2) = 0.$$

So $d_0 = 0$ if and only if

$$h_0^2 - h_1^2 + \cdots + (-1)^{n-1} h_{n-1}^2 \neq 0.$$

COROLLARY. *If* $h_i = \mathrm{const}(i = 0, 1, \ldots, n-1)$ *and* n *is odd, or if* $h_0 \leqslant h_1 \leqslant \cdots \leqslant h_{n-1}$ *(or* $h_0 \geqslant h_1 \geqslant \cdots \geqslant h_{n-1}$*) and there is at least one inequality, the solution of the above problem uniquely exists.*

5. VARIATIONAL PROPERTIES

THEOREM 5.7. *Let* $s(x) \in Sp(\Delta, k)$ *be the solution of the* $(k, k-2)$ $(k = 2, 3)$ *problem and let* $f(x) \in PC_\Delta^k[a, b]$ *be an arbitrary function. Then the first integration relationships hold*:

local:

$$J_i[f^{(k-1)}] = J_i[s^{(k-1)}] + J_i[(f-s)^{(k-1)}] \qquad (i = 0, 1, \ldots, n-1),$$

global:

$$J[f^{(k-1)}] = J[s^{(k-1)}] + J[(f-s)^{(k-1)}] \qquad (k = 2, 3).$$

We note

$$J_i[(f-s)^{(k-1)}] = J_i[f^{(k-1)}] - J_i[s^{(k-1)}] - 2I[(f-s)^{(k-1)}, s^{(k-1)}],$$

where

$$\begin{aligned} I[(f-s)^{(k-1)}, s^{(k-1)}] &= \int_{x_i}^{x_{i+1}} [f^{(k-1)}(x) + f^{(k-1)}(x_i + x_{i+1} - x) \\ &\quad - s^{(k-1)}(x) - s^{(k-1)}(x_i + x_{i+1} - x)] \\ &\quad \times [s^{(k-1)}(x) + s^{(k-1)}(x_i + x_{i+1} - x)]\, dx. \end{aligned} \tag{5.1}$$

By integration by parts $I[(f-s)^{(k-1)}, s^{(k-1)}] = 0$.

THEOREM 5.8. *Under the conditions of Theorem 5.7, we have the following relationships:*

local:

$$J_i[f^{(k-1)}] \geqslant J_i[s^{(k-1)}] \qquad (i = 0, 1, \dots, n-1),$$

global:

$$J[f^{(k-1)}] \geqslant J[s^{(k-1)}] \qquad (k = 2, 3).$$

Notice $J_i[(f-s)^{(k-1)}] \geqslant 0$, $J[(f-s)^{(k-1)}] \geqslant 0$, and apply Theorem 5.7 to prove Theorem 5.8.

Remark. By Theorems 3.1 and 3.2 we know that the minimization for problem $J_i[f^{(k-1)}]$, $J[f^{(k-1)}]$ on $PC_\Delta^k[a, b]$ has no unique solution. However, there is one on $Sp(\Delta, k)$ which satisfies the end point conditions.

THEOREM 5.9. *Let $f(x) \in PC_\Delta^k[a, b]$. Let $s_f(x) \in Sp(\Delta, k)$ be the solution of the $(k, k-2)$ interpolation problem for $f(x)$, and let $s(x) \in Sp(\Delta, k)$ be an arbitrary spline. The following conditions hold:*

local*:*

$$J_i[(f-s_f)^{(k-1)}] \leqslant J_i[(f-s)^{(k-1)}] \qquad (i = 0, 1, \dots, n-1),$$

global:

$$J[(f-s_f)^{(k-1)}] \leqslant J[(f-s)^{(k-1)}] \qquad (k = 2, 3). \tag{5.2}$$

If $s(x)$ and $s_f(x)$ have the same end point conditions, equality holds in (5.2) only if $s_f(x) \equiv s(x)$.

Proof. We have

$$\begin{aligned} J_i[(f-s)^{(k-1)}] &= J_i[(f-s_f)^{(k-1)}] + J_i[(s_f - s)^{(k-1)}] \\ &\quad + 2I[(f-s_f)^{(k-1)}, (s_f - s)^{(k-1)}], \end{aligned}$$

where $I[(f-s_f)^{(k-1)}, (s_f-s)^{(k-1)}]$ is similar to (5.1). By integration by parts:

$$I[(f-s_f)^{(k-1)}, (s_f-s)^{(k-1)}]=0.$$

Therefore,

$$J_i[(f-s)^{(k-1)}]=J_i[(f-s_f)^{(k-1)}]+J_i[(s_f-s)^{(k-1)}].$$

But

$$J_i[(s_f-s)^{(k-1)}]\geqslant 0, \qquad J[(s_f-s)^{(k-1)}]\geqslant 0,$$

which prove the theorem.

THEOREM 5.10. *Let $f(x)\in PC^k_\Delta[a,b]$. Let $s_f(x)\in Sp(\Delta,k)$ be the solution of the $(k,k-2)$ interpolation problem for $f(x)$. The second integration relationships hold:*

local:

$$J_i[(f-s_f)^{(k-1)}]=-L_i[(f-s_f)^{(k-2)},f^{(k)}] \qquad (i=0,1,...,n-1),$$

global:

$$J[(f-s_f)^{(k-1)}]=-L[(f-s_f)^{(k-2)},f^{(k)}] \qquad (k=2,3),$$

where

$$\begin{aligned} L_i&[(f-s_f)^{(k-1)},f^{(k)}]\\ &=\int_{x_i}^{x_{i+1}}[f^{(k-2)}(x)-s_f^{(k-2)}(x)-f^{(k-2)}(x_i+x_{i+1}-x)\\ &\quad+s_f^{(k-2)}(x_i+x_{i+1}-x)]\\ &\quad\times[f^{(k)}(x)-f^{(k)}(x_i+x_{i+1}-x)]\,dx \qquad (i=0,1,...,n-1),\\ L&[(f-s_f)^{(k-2)},f^{(k)}]\\ &=\sum_{i=0}^{n-1}L_i[(f-s_f)^{(k-2)},f^{(k)}] \qquad (k=2,3). \end{aligned}$$

Proof is by integration by parts.

6. Error Bounds

Let $\|f^{(j)}\|=\max_{x\in[a,b]}|f^{(j)}(x)|$, $V_a^b(f^{(j)})=\int_a^b|df^{(j)}|$, $M_j=\|f^{(j)}\|+V_a^b(f^{(j)})$, $h=\max_{0\leqslant i\leqslant n-1}h_i$, $\max h_i/\min h_i\leqslant\beta_0$, $(\sum_{j=1}^{n-1}|h_j-h_{j+1}|)/h\leqslant\beta_1$.

$\max_{0 \leqslant m \leqslant n-1}(\max_{m<i}(\sum_{j=m+1}^{i} |h_j - h_{j+1}|)/h_m, \ \max_{i<m}(\sum_{j=i+1}^{m} |h_j - h_{j+1}|)/h_m) \leqslant \beta_2$, $R(f, x) = f(x) - s_f(x)$.

By Peano's theorem we obtain

$$R(f, x) = \int_a^b R_x\{(x-t)_+^2/2!\} f'''(t)\, dt,$$

where $R_x\{(x-t)_+^2/2!\}$ is the remainder of the (2, 0) interpolation problem for the function $(x-t)_+^2/2!$ of argument x. We can show [2]

$$R_x\{(x-t)_+^2/2!\} = R_t\{(x-t)_+^2/2!\}.$$

Therefore,

$$R^{(j)}(f, x) = R_t^{(-1)}\{(x-t)_+^{2-j}/(2-j)!\}|_{t=b} f'''(b) - \int_a^b R_u^{(-1)}\{(x-u)_+^{2-j}/(2-j)!\}\, df'''(u) \qquad (j = 0, 1, 2),$$

where

$$R_t^{(-1)}\{(x-t)_+^j/j!\} = \int_a^t R_u\{(x-u)_+^j/j!\}\, du.$$

Therefore,

$$|R^{(j)}(f, x)| \leqslant \max_{a \leqslant t, x \leqslant b} |R_t^{(-1)}\{(x-t)_+^{2-j}/(2-j)!\}|\, M_3 \qquad (j = 0, 1, 2).$$

By Theorem 3.3,

$$\begin{aligned} R_t\{(x-t)_+^2/2!\} &= 0, \qquad a \leqslant t \leqslant x_m, \\ &= (x-t)^2/2 - (x-x_m)(x_{m+1}-t)[(t-x_m)(x-x_m-2h_m) \\ &\quad + (x-x_m)\, h_m]/2h_m^2, \qquad x_m \leqslant t \leqslant x, \\ &= -(x-x_m)(x_{m+1}-t)[(x-x_m-2h_m)(t-x_m) + (x-x_m)\, h_m]/2h_m^2, \\ &\qquad\qquad x \leqslant t \leqslant x_{m+1}, \\ &= (-1)^{i-m-1}(x-x_m)(x-x_m-h_m)(t-x_i)(x_{i+1}-t)/h_m h_i, \\ &\quad x_i \leqslant t \leqslant x_{i+1}, \ i = m+1, \ldots, n-1. \end{aligned}$$

Therefore, we obtain Theorem 6.11.

THEOREM 6.11. *If $f(x) \in C^3[a,b]$ with $f^{(3)}$ of bounded variation, then for the solution of the* (2, 0) *problem and its derivatives we have*

$$\|R^{(j)}(f,x)\| \leqslant c_j M_3 h^{3-j} \qquad (j=0,1,2), \tag{6.1}$$

where $c_0 = (4+3\beta_1)/36$, $c_1 = (4+3\beta_1)/18$, $c_2 = (5+\beta_0+2\beta_2)/6$.

COROLLARY 6.11. *Under the conditions of Theorem* 6.11, *if $\{h_i\}$ is a monotone sequence then* (6.1) *holds, where* $c_0 = 7/36$, $c_1 = 7/18$, $c_2 = (5+3\beta_0)/6$.

If we take $s_f'(x)$ as the solution of the (2, 0) problem for $f'(x)$, we obtain Theorem 6.12 directly from Theorem 6.11.

THEOREM 6.12. *If $f(x) \in C^4[a,b]$ with $f^{(4)}$ of bounded variation, then for the solution of the* (3.1) *problem and its derivatives we have*

$$\|R(f,x)\| \leqslant c_0(b-a)\, M_4 h^3,$$

$$\|R^{(j+1)}(f,x)\| \leqslant c_j M_4 h^{3-j} \qquad (j=0,1,2),$$

where c_0, c_1, c_2 are the same as in Theorem 6.11.

REFERENCES

1. J. H. AHLBERG, E. N. NILSON, AND J. L. WALSH, "The Theory of Splines and Their Applications," Academic Press, New York, 1967.
2. Y. LI AND D. QI, "Spline Function Methods," Chinese Science Publ., Beijing, 1979. [Chinese]
3. A. SHARMA AND J. TZIMBALARIO, Quadratic splines, *J. Approx. Theory* **19** (1977), 186–193.
4. C. DE BOOR, Quadratic spline interpolation and the sharpness of Lebesgue's inequality, *J. Approx. Theory* **17** (1976), 348–358.
5. J. W. DANIEL, Constrained approximation and Hermite interpolation with smooth parabolic splines: some negative results, *J. Approx. Theory* **17** (1976), 135–149.
6. G. MEINARDUS AND G. D. TAYLOR, Periodic quadratic spline interpolant of minimal norm, *J. Approx. Theory* **23** (1978), 137–141.
7. E. NEUMAN, Determination of a quadratic spline function with given value of the integrals in subintervals, *Zastos. Mat.* **16** (1980), 681–689.
8. E. NEUMAN, Quadratic splines and histospline projections, *J. Approx. Theory* **29** (1980), 297–304.